

## NONLOCAL FORMULATION OF THE EVOLUTION OF DAMAGE IN A ONE-DIMENSIONAL CONFIGURATION

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(Received 10 March 1992; in revised form 23 October 1992)

**Abstract**—The constitutive modelling of distributed damage evolution with strain-softening effects causes serious problems when local concepts are employed: numerical approximations obtained by finite element techniques show unlimited localization of deformation and energy dissipation decreases to unrealistic values when element meshes are refined. To overcome these problems nonlocal formulations have been introduced; a particular continuum damage example is presented which allows an analytical elaboration in the one-dimensional case. For a bar in uniaxial tension the exact solution is derived and the physical implications are examined.

### 1. INTRODUCTION

The modelling of strain-softening behaviour (decreasing stress at increasing strain) of brittle materials like concrete, rock, glass and ceramics, using local continuum mechanics quantities, leads to spurious solutions. A one-dimensional consideration suffices to expose the difficulties.

Suppose that a bar (with initial reference length  $L$  and uniform cross-sectional area  $A$ ) is submitted to a uniaxial tensile stress  $\sigma(t)$ . The (pseudo-) time variable  $t$  does not indicate explicit time dependence but will only serve as a parameter defining subsequent states of the bar. The elongation is denoted by  $\Delta L(t)$  and the average strain  $\epsilon_a(t)$  is defined as

$$\epsilon_a(t) = \frac{\Delta L(t)}{L}. \tag{1}$$

It is assumed that experimental investigations of the mechanical behaviour of the bar under tensile loading resulted in a relationship between stress and average strain as visualized in Fig. 1. The ascending (elastic) and descending (strain-softening) branches in the graph can be formulated as

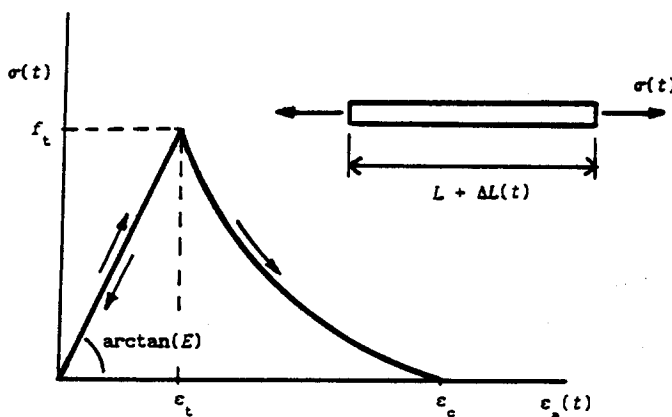


Fig. 1. Bar under tensile loading.

$$\begin{aligned}\sigma(t) &= E\varepsilon_a(t) & \text{if } \varepsilon_a(t) \leq \varepsilon_t = f_t/E, \\ \sigma(t) &= S(\varepsilon_a(t)) & \text{if } \varepsilon_a(t) \geq \varepsilon_t = f_t/E,\end{aligned}\quad (2)$$

with Young's modulus  $E$  and tensile strength  $f_t$ ; complete loss of stiffness occurs at the critical strain  $\varepsilon_c$ . In a classical consideration the validity of (2) is accepted for every material point  $x \in [-L/2, L/2]$  of the bar: it is assumed that the local constitutive description can be derived from (2), replacing the average strain  $\varepsilon_a(t)$  by the local strain  $\varepsilon(x, t)$ . It is this very assumption that causes the well-known ambiguity of equilibrium solutions for an initially homogeneous bar (Bazant, 1986, 1988; Schreyer and Chen, 1986; De Borst and Mühlhaus, 1991): when after continuous straining  $\sigma(t)$  reaches the maximum value  $f_t$ , the state of a separate material point may follow the softening branch, but may descend along the elastic branch as well. If the loading process is performed on a bar with imperfections, deformation will localize in the weakest cross-section in combination with vanishing energy dissipation. Experiences (Pietruszczak and Mroz, 1981; Bazant *et al.*, 1987; Needleman, 1988) with finite element calculations based on local concepts for strain-softening materials show similar unrealistic phenomena. The consequences of conventional local constitutive modelling are physically unacceptable.

To avoid the problems as outlined above, a number of distinguishable techniques can be employed to regularize the modelling of reality (Simo, 1988). To confine the actual considerations, attention will be focussed on nonlocal formulations of the constitutive behaviour, being the particular subject of the present research. A variety of nonlocal approaches has been proposed during the last decade. From this variety a selection of representative contributions in the literature (Coleman and Hodgdon, 1985; Triantafyllidis and Aifantis, 1986; Pijaudier-Cabot and Bazant, 1987; Saouridis and Mazars, 1988; Schreyer, 1988; Bazant and Zubelewicz, 1988; Schreyer, 1990) will briefly be discussed in the concluding section of this paper, to explain the distinction between these references and the procedure to be developed here. As a point of departure, a continuum damage formulation will be chosen. In the stress-strain relationship a damage variable  $D(x, t)$  is introduced to account for the effect of distributed material degeneration. This part of the constitutive description is kept local, however, the evolution equation for the damage variable  $D(x, t)$  is presented in a nonlocal way.

## 2. NONLOCAL CONTINUUM DAMAGE

The theory of continuum damage mechanics (Kachanov, 1986; Simo and Ju, 1987; Lemaitre and Chaboche, 1988) is based on the incorporation of internal variables into the material characterization to describe the local deterioration. By identification of these damage variables and prescribing the evolutions by suitable models the influence of defects on the material behaviour can be predicted. In this paper considerations will be confined to linearly elastic material affected by one damage quantity  $D(x, t)$  as the only relevant internal variable. This damage  $D(x, t)$  is a monotonically nondecreasing scalar  $0 \leq D(x, t) \leq 1$ , expressing the local instantaneous level of material degradation; initially undamaged material is characterized by  $D = 0$  while  $D = 1$  indicates complete loss of stiffness. The principle of strain equivalence, applied to the uniaxially loaded bar, leads to

$$\sigma(t) = [1 - D(x, t)]E\varepsilon(x, t), \quad (3)$$

where the effective elastic modulus can be recognized as the product of the reduction factor  $1 - D$  and Young's modulus  $E$ . The constitutive modelling (3) must be completed by the specification of a damage criterion and a damage evolution law. On behalf of the damage criterion a threshold functional of the strain, the damage equivalent strain  $\varepsilon_d(x, t)$ , is introduced, defined locally by the expression

$$\varepsilon_d(x, t) = \text{Max} \{ \varepsilon_t; \varepsilon(x, \tau) \mid \tau \leq t \}, \quad (4)$$

with  $\varepsilon_c$  the initial threshold value, a material property applied previously in (2). The damage criterion controls the damage evolution, which generally satisfies the equations

$$\begin{aligned} \dot{D}(x, t) &> 0 && \text{if } \varepsilon(x, t) = \varepsilon_d(x, t) \quad \text{and if } \dot{\varepsilon}(x, t) > 0, \\ \dot{D}(x, t) &= 0 && \text{if } \varepsilon(x, t) < \varepsilon_d(x, t) \quad \text{or if } \dot{\varepsilon}(x, t) \leq 0, \end{aligned} \tag{5}$$

where the dot superscript denotes differentiation to the time variable. The nonlocal damage evolution equation is derived from specific considerations with respect to a reformulation of relationship (3) :

$$D(x, t) = \frac{\varepsilon(x, t) - \frac{\sigma(t)}{E}}{\varepsilon(x, t)}. \tag{6}$$

This particular equation describes the damage as the (weighted) difference of the actual strain  $\varepsilon(x, t)$  of the damaged material and a fictitious strain  $\sigma(t)/E$  as if the material were undamaged. In the case of localization of damage in a zone of the bar, significant growth of damage will occur only in this zone, while the rest of the bar remains virtually unaffected. This suggests that to obtain the fictitious strain quantity  $\sigma(t)/E$  necessary to compute the damage according to (6), the damage evolution equation can be given by an adequate evaluation of the strain information provided by the material of the bar itself. Entirely in harmony with this suggestion the following proposition is made for the damage evolution :

$$\begin{aligned} D(x, t) &= \int_{-\infty}^{\infty} W(s, t) \frac{\|\varepsilon(x, t) - \varepsilon(x+s, t)\|}{\varepsilon(x, t)} ds && \text{if } \varepsilon(x, t) = \varepsilon_d(x, t) \quad \text{and if } \dot{\varepsilon}(x, t) > 0, \\ D(x, t) &= \text{Max} \{D(x, \tau) \mid \tau \leq t\} && \text{if } \varepsilon(x, t) < \varepsilon_d(x, t) \quad \text{or if } \dot{\varepsilon}(x, t) \leq 0. \end{aligned} \tag{7}$$

In this formulation the weighting function  $W(s, t)$  is introduced, even in  $s$  and positive, dependent on process conditions and material parameters. The dummy variable  $s$  is used as the integration variable, denoting the material points of the bar. As the integration domain  $\langle -\infty, \infty \rangle$  exceeds the length of the bar, the strain field is fictitiously extended by  $\varepsilon(x, t) = \sigma(t)/E$  for  $\|x\| > L/2$ . This extension allows for the description of homogeneous deformation [process with prescribed strain as performed by Mazars *et al.* (1988)], if a suitable choice for  $W(s, t)$  is made. However, in this paper interest will be concentrated on localization phenomena.

### 3. LOCALIZATION

The specification of damage evolution according to (7) does not necessarily lead to the desired limitation of deformation localization unless a proper expression for  $W(s, t)$  is substituted. On behalf of the transparency of the elaboration some simplifying assumptions and approximations are presented, especially for the investigation of localization.

It will be assumed that the weighting function  $W$  is independent of the time  $t$  while the range of significance with respect to the coordinate  $s$  is relatively small [see e.g. Bazant and Chang (1984)]:  $W(s) = 0$  for  $\|s\| > \Delta$  with  $\Delta \ll L$ . Then the damage evolution (7) can be approximated by

$$\begin{aligned} D(x, t) &= \frac{1}{\varepsilon(x, t)} \int_{-\Delta}^{\Delta} W(s) \left\| s \frac{\partial \varepsilon}{\partial s} \right\|_x ds = \frac{m}{\varepsilon(x, t)} \left\| \frac{\partial \varepsilon}{\partial x} \right\| \\ &&& \text{if } \varepsilon(x, t) = \varepsilon_d(x, t) \quad \text{and if } \dot{\varepsilon}(x, t) > 0, \\ D(x, t) &= \text{Max} \{D(x, \tau) \mid \tau \leq t\} && \text{if } \varepsilon(x, t) < \varepsilon_d(x, t) \quad \text{or if } \dot{\varepsilon}(x, t) \leq 0 \end{aligned} \tag{8}$$

where the moment integral  $m$  of  $W(s)$  is defined by

$$m = \int_{-\Delta}^{\Delta} W(s) \|s\| ds. \quad (9)$$

Higher order approximations can be achieved by the extension of (8) with additional terms of the Taylor series expansion of the strain  $\varepsilon$ . The actual formulation (8), however, suffices to demonstrate the consequences of the nonlocal approach followed in this paper. In fact, the combination of the relationships (3), (4) and (8) will be considered as a refoundation of the theoretical problem formulation.

Attention is focussed on a tensile bar with a continuous increase in deterioration. The point  $x = 0$  is arbitrarily indicated as the centre of symmetry with respect to the variables to be determined. In the case of a single localization the deformation will be concentrated in the vicinity of  $x = 0$ . The following elaborations are confined to locations with  $x \geq 0$  where the strain gradient is assumed to be nonpositive. Substitution of (8) into (3) yields

$$\begin{aligned} \sigma(t) &= E \left[ \varepsilon(x, t) + m \frac{\partial \varepsilon}{\partial x} \right] \quad \text{if } \varepsilon(x, t) = \varepsilon_d(x, t) \quad \text{and if } \dot{\varepsilon}(x, t) > 0, \\ \sigma(t) &= E\varepsilon(x, t)[1 - D(x, t)] \quad \text{if } \varepsilon(x, t) < \varepsilon_d(x, t) \quad \text{or if } \dot{\varepsilon}(x, t) \leq 0. \end{aligned} \quad (10)$$

Based on (10) the strain field  $\varepsilon(x, t)$  in the bar can be expressed in the tensile stress  $\sigma(t)$ , resulting in

$$\begin{aligned} \varepsilon(x, t) &= \frac{\sigma(t)}{E} + \left[ \varepsilon_0(t) - \frac{\sigma(t)}{E} \right] e^{-x/m} \quad \text{while } \varepsilon(x, t) = \varepsilon_d(x, t) \quad \text{and while } \dot{\varepsilon}(x, t) > 0, \\ \varepsilon(x, t) &= \frac{\sigma(t)}{E[1 - D(x, t)]} \quad \text{while } \varepsilon(x, t) < \varepsilon_d(x, t) \quad \text{or while } \dot{\varepsilon}(x, t) \leq 0, \end{aligned} \quad (11)$$

where  $\varepsilon_0(t)$  symbolizes the strain in the centre  $x = 0$  of the localization. By differentiation of (11) with respect to  $x$  and substitution of the result into (8) it can easily be shown that the damage  $D(x, t)$  is monotonously nonincreasing with the distance to the central point.

To determine the strain field  $\varepsilon(x, t)$  as an explicit function of the stress  $\sigma(t)$ , the relationship between the increasing central strain  $\varepsilon_0(t)$  and the stress  $\sigma(t)$  to be considered as a characteristic material feature, should be specified. In favour of mathematical convenience a simple power-law is introduced to describe the softening curve

$$\frac{\sigma(t)}{f_t} = \left[ \frac{\varepsilon_c - \varepsilon_0(t)}{\varepsilon_c - \varepsilon_t} \right]^r \quad (12)$$

for  $\varepsilon_0(t) \geq \varepsilon_t$ , with  $\varepsilon_t$  the strain at damage initiation,  $f_t$  the associated tensile strength,  $\varepsilon_c$  the critical value of  $\varepsilon_0$  at complete loss of stiffness and  $r$  a positive constant. This power-law can be substituted into (11). It is required then to distinguish the localization zone  $x < z(t)$  of the bar with damage increase from the region where elastic relaxation occurs. The actual value of  $z(t)$  can be established from the change of sign of  $\partial \varepsilon / \partial \sigma$ . The analysis of the behaviour of the bar will be continued subsequently for  $r = 1$ , for  $0 < r < 1$  and for  $r > 1$ . Evaluation of the strain and damage distributions should definitely provide understanding of the relevant local phenomena. Besides, integral quantities such as the energy dissipation and the average strain will be computed.

For  $r = 1$  (linear softening) it can be derived from (12) that

$$\varepsilon_0(t) = \varepsilon_c - [\varepsilon_c - \varepsilon_t] \frac{\sigma(t)}{f_t} \tag{13}$$

The strain field during softening can be constructed by substitution of (13) into (11). This leads to the result

$$\begin{aligned} \varepsilon(x, t) &= \varepsilon_t \left[ \frac{\sigma(t)}{f_t} + \left[ 1 - \frac{\sigma(t)}{f_t} \right] e^{[z-x]/m} \right] \quad \text{if } x < z, \\ \varepsilon(x, t) &= \varepsilon_t \left[ \frac{\sigma(t)}{f_t} \right] \quad \text{if } x \geq z, \end{aligned} \tag{14}$$

where (half) the localization width  $z$  is given by

$$z = m \cdot \log \left[ \frac{\varepsilon_c}{\varepsilon_t} \right]. \tag{15}$$

It can be concluded that the damage accumulates in an invariant zone of the tensile bar while outside of this zone the material obviously remains fully undamaged. The width  $2z$  of the localization zone depends on the parameter  $m$  defined by (9), and on the scaled critical strain  $\varepsilon_c/\varepsilon_t$ . For a particular ratio  $\varepsilon_c/\varepsilon_t = 5.0$  (which is a reasonable value for brittle materials) and thus  $z/m \approx 1.6$ , stress is plotted versus strain for various cross-sections of the bar in Fig. 2. It is easily verified that during strain-softening  $D(x, t)$  satisfies

$$\begin{aligned} D(x, t) &= \frac{\left[ 1 - \frac{\sigma(t)}{f_t} \right] e^{[z-x]/m}}{\frac{\sigma(t)}{f_t} + \left[ 1 - \frac{\sigma(t)}{f_t} \right] e^{[z-x]/m}} \quad \text{if } x < z, \\ D(x, t) &= 0 \quad \text{if } x \geq z. \end{aligned} \tag{16}$$

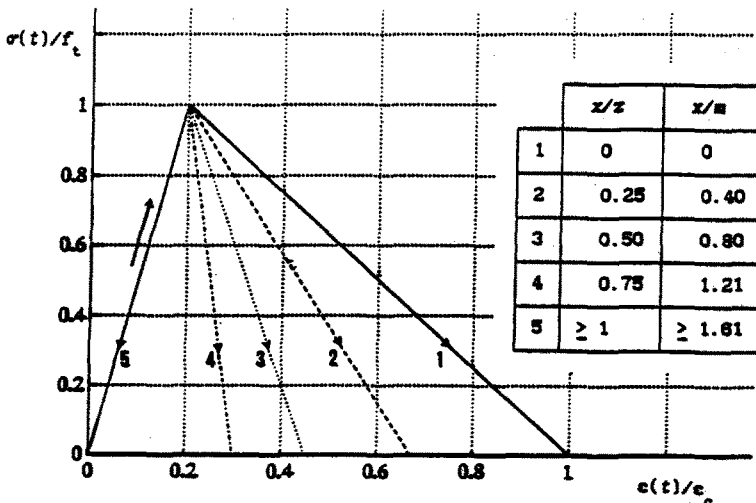


Fig. 2. Stress versus strain for  $r = 1$  and  $\varepsilon_c/\varepsilon_t = 5.0$ .

In Fig. 3 this result is displayed ; for different load levels the damage is shown as a function of the scaled coordinate  $x/z$ . The damage increases while the tensile stress  $\sigma(t)$  decreases from  $f_t$  to zero. At complete rupture of the bar the damage equals 1 for all  $x$  in the localization zone. With the strain field available for all values of  $\sigma(t)$  until rupture, the total energy  $G_f$  dissipated per unit of cross-sectional area (energy release rate) can be calculated, eventually leading to the simple expression

$$G_f = m f_t \varepsilon_t [e^{z/m} - 1] = m f_t [\varepsilon_c - \varepsilon_t]. \tag{17}$$

By integration of  $\varepsilon(x, t)$  in (14) over the domain  $[0, L/2]$  of  $x$ , (half) the total elongation during softening can be determined as a function of  $\sigma(t)$ . Application of (1) results in the average strain  $\varepsilon_a(t)$  according to

$$\varepsilon_a(t) = \frac{\sigma(t)}{f_t} \left[ \varepsilon_t - \frac{2m}{L} [\varepsilon_c - \varepsilon_t] \right] + \frac{2m}{L} [\varepsilon_c - \varepsilon_t]. \tag{18}$$

The size effect is clearly represented by the ratio  $m/L$ ; the descending branch of the stress versus average strain curve starts to snap back if

$$L > 2m \left[ \frac{\varepsilon_c}{\varepsilon_t} - 1 \right]. \tag{19}$$

It can be concluded that strain softening behaviour in a global sense can only be observed for a sufficiently small length  $L$  of the tensile bar. For  $L$  satisfying (19) a hypothetical experiment with continuous straining results in purely brittle failure.

The procedure outlined above can be followed in an analogous way in case the exponent  $r$  in (12) is unequal to 1. However, the analytical elaboration is considerably more complicated and the results are less transparent. The most interesting quantity appears to be the (half) width  $z(t)$  of the zone with deformation localization. The calculation of  $z(t)$ , based on (11) in combination with (12), leads to

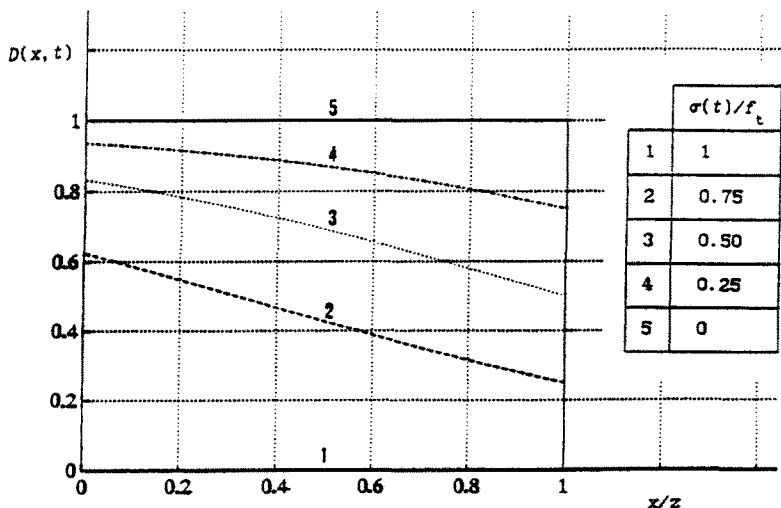


Fig. 3. Damage evolution for  $r = 1$  and  $\varepsilon_c/\varepsilon_t = 5.0$ .

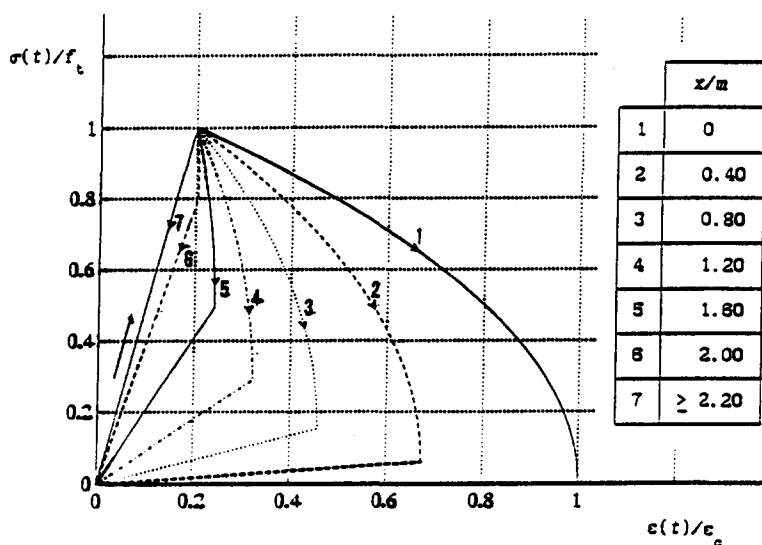


Fig. 4. Stress versus strain for  $r = 1/2$  and  $\epsilon_c/\epsilon_t = 5.0$ .

$$z(t) = m^\epsilon \log \left[ 1 + \frac{1}{r} \left[ \frac{\epsilon_c}{\epsilon_t} - 1 \right] \left[ \frac{\sigma(t)}{f_t} \right]^{[(1/r)-1]} \right] \quad \text{if } 0 < r < 1,$$

$$z = m^\epsilon \log \left[ 1 + \frac{1}{r} \left[ \frac{\epsilon_c}{\epsilon_t} - 1 \right] \right] \quad \text{if } r \geq 1. \quad (20)$$

This means that for  $0 < r < 1$  the localization zone decreases continuously and vanishes in the final state of fracture. For  $r > 1$  the localization zone remains constant, completely comparable with the situation for  $r = 1$ . The relation between stress and strain is visualized in Fig. 4 for  $r = 1/2$  (the reducing zone of localization can clearly be observed) and in Fig. 5 for  $r = 2$ , both with  $\epsilon_c/\epsilon_t = 5.0$ . For any value of the exponent  $r$ , a nonzero energy dissipation can be determined. If  $0 < r < 1$ , the elongation at failure disappears due to fully elastic relaxation in the whole domain of the bar.

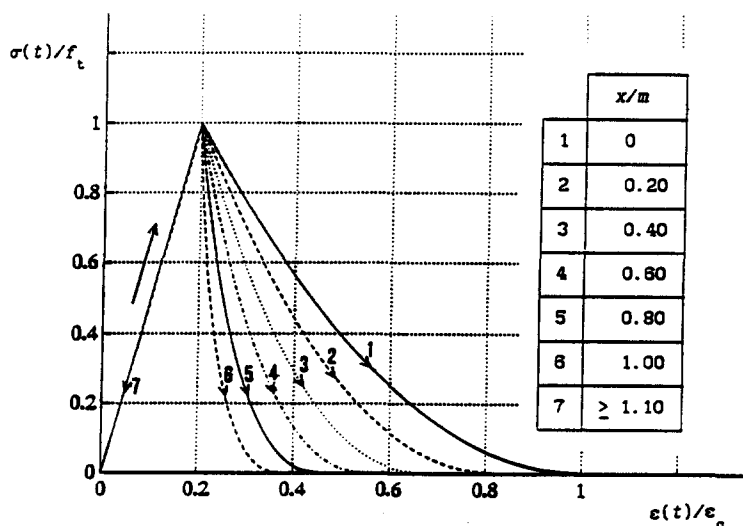


Fig. 5. Stress versus strain for  $r = 2$  and  $\epsilon_c/\epsilon_t = 5.0$ .

## 4. CONCLUDING REMARKS

In the foregoing, the localization of deformation in a one-dimensional configuration has been discussed. The localization is directly caused by softening, the type of localization has theoretically been investigated in dependence on the material characteristics. A proper nonlocal constitutive continuum damage formulation, presented here by incorporation of the strain gradient into the damage evolution equation, proves to be capable of deriving realistic results. Indeed an unlimited localization is avoided, a finite energy dissipation has been observed and scale effects could be shown. To compare the method developed here, to contributions in the literature, a representative selection will be considered in the following discussion.

Pijaudier-Cabot and Bazant (1987) introduced a nonlocal damage theory, where, instead of the basic damage equation (8) applied before, damage has been defined as a function of the nonlocal damage energy release rate which is determined by a weighted integration of the local rate over the volume of the configuration. Even in the uniaxial situation an analytical solution could not be derived and, without detracting from the importance of their approach, the precise consequences of the mathematical modelling are less transparent than in the case presented here. Besides, the evaluation of weighted integral quantities in the vicinity of the boundary of the volume (unilateral reduction of the integration domain) may induce unrealistic phenomena in the solution of the system of equations. This can be shown by the numerical analysis of a tensile bar (with a very small imperfection to initiate the localization of deformation): the centre of the localization band may, under particular circumstances, move in the axial direction through the material. It is beyond the scope of this paper to enter into details on this area. An approach, closely related to the procedure proposed in the latter reference, has been elaborated by Saouridis and Mazars (1988). They considered the damage as a function of a nonlocal damage equivalent strain, fully comparable to the nonlocal energy release rate mentioned above and similarly defined. Due to this similarity, the same comments as before are relevant.

Schreyer (1988) postulated a constitutive modelling theory, where the gradient of the damage parameter was introduced as a nonlocal variable in the damage evolution equation. For a bar in tension an analytical solution could be derived. For any value of the length of the bar his formulation leads inevitably to snap-back in the load versus displacement curve. It is generally accepted that for a sufficiently small length of the bar the snap-back behaviour disappears, indicating a deficiency in the assumptions made in the development of the theory. Moreover, in a more general context the implications for a finite element implementation (displacement method) should be considered. It can be remarked that the introduction of the gradient of damage (being a derivative quantity to be calculated from the constitutive equations) is less efficient to perform than the account for the gradient of strain (as used in (8) here), which is directly expressible in the nodal point displacements.

Nonlocal procedures, occurring in the modelling theories for softening material behaviour, are certainly not restricted to the elaboration of explicit damage concepts. Bazant and Zubelewicz (1988) presented a fully nonlocal approach in a one-dimensional formulation: the total strain was decomposed in a local and a nonlocal component, two associated stresses were introduced (a linear combination leads to the total actual stress) while the strain and stress quantities were constitutively coupled by two different incremental elastic moduli. The nonlocal strain was defined as the sum of the local strain and a factorized second order strain gradient. The "post-peak" behaviour was supposed to be the combined response of a segment of the material configuration (with approximately constant length emanating from stability considerations) with increasing deformation and the rest of the material, unloading elastically. Eventually this theory results in a sixth order differential equation which, in the simplest case, can be solved analytically. How to perform a generalization of the theory to more-dimensional situations is a serious problem which is left out of consideration.

Triantafyllidis and Aifantis (1986) focussed their attention on the localization of deformation for hyperelastic materials with softening at large levels of strain. They proposed to enrich the strain energy function (and consequently the stress-strain relationship) by incorporation of the second order gradient of the displacement (first order gradient of



strain). Also in the present paper the first order gradient of strain represents the nonlocal term, however this is the only correspondence as the further elaboration differs completely due to the dominating effect of history dependence in continuum damage mechanics here. From the classical point of view, history dependence is the essential feature in the field of (elasto-, visco- or elastovisco-) plasticity, observable by strain hardening or, more important in the actual context, softening. The development of shear bands in ductile, rigid plastic materials has been analysed by Coleman and Hodgdon (1985). In their work the introduction of nonlocal concepts was required as well for an adequate modelling of the structural behaviour. Based upon heuristic reasoning, they modified the yield function by incorporation of the second order spatial derivative of the equivalent plastic strain, an internal variable comparable to damage. After a complicated calculation analytical results have been derived. With respect to the present paper the nature of the nonlocal variable and the order of spatial derivative in the definition of that variable are different. The work of Schreyer (1990) also involves the modelling of localization in the theory of plasticity. The stress-strain curve was assumed to be piecewise linear for strain hardening and softening. He introduced the nonlocal feature into the constitutive description by the assumption that the yield stress depends on an invariant of the gradient of plastic strain. Analytical solutions have been obtained for the case that this functional dependence is quadratic. An essential result is that the softening zone evolves from a material point while the theory presented here predicts a constant or decreasing zone. Experimental observations should be able to resolve this rather interesting difference.

It can be stated that the approach used in this research to describe localization due to softening behaviour should certainly be considered as an alternative method with particular characteristics and merits. Continuum damage mechanics offered the fundamental basis to develop and investigate the actual theory. Adaptations of the postulated damage criterion and the damage evolution law (here defined for the one-dimensional case), by introduction of a suitable scalar measure of the components of the strain tensor, extend the theory presented to more-dimensional situations. The appendix at the end of this paper gives an outline of the generalization to three-dimensional configurations. However, an analytical elaboration will generally not be possible and application of numerical procedures is necessary to find solutions. The considerations in this paper contribute to a better understanding of these solutions.

#### REFERENCES

- Bazant, Z. P. (1986). Mechanics of distributed cracking. *Appl. Mech. Rev. ASME* **39**, 675–705.
- Bazant, Z. P. (1988). Stable states and stable paths of propagation of damage zones and interactive fractures. In *Proc. France-US Workshop on Strain Localization and Size Effect due to Cracking and Damage*, Cachan, France (Edited by J. Mazars and Z. P. Bazant), pp. 183–206. Elsevier, London.
- Bazant, Z. P. and Chang, T. P. (1984). Instability of nonlocal continuum and strain averaging. *J. Engng Mech. ASCE* **110**, 1441–1450.
- Bazant, Z. P., Lin, F. B. and Pijaudier-Cabot, G. (1987). Yield limit degradation: nonlocal continuum model with local strain. In *Proc. Int. Conf. on Computational Plasticity*, Barcelona (Edited by E. Onate, R. Owen and E. Hinton), pp. 1757–1779. Pineridge Press, Swansea.
- Bazant, Z. P. and Zubelewicz, A. (1988). Strain-softening bar and beam: exact non-local solution. *Int. J. Solids Structures* **24**, 659–673.
- Coleman, B. D. and Hodgdon, M. L. (1985). On shear bands in ductile materials. *Arch. Rat. Mech. Anal.* **90**, 219–247.
- De Borst, R. and Mühlhaus, H. B. (1991). Continuum models for discontinuous media. In *Proc. Int. RILEM/ESIS Conf. on Fracture Processes in Concrete, Rock and Ceramics*, Noordwijk, The Netherlands (Edited by J. G. M. Van Mier, J. G. Rots and A. Bakker), pp. 601–618. E & FN Spon/Chapman & Hall, London.
- Kachanov, L. M. (1986). *Introduction to Continuum Damage Mechanics*. Nijhoff, Dordrecht, The Netherlands.
- Lemaitre, J. and Chaboche, J. L. (1988). *Mechanique des Materiaux Solides*. Dunod, Paris.
- Mazars, J., Ramtani, S. and Berthaud, Y. (1988). An experimental procedure to delocalize tensile failure and to identify the unilateral effect of distributed damage. In *Proc. France-US Workshop on Strain Localization and Size Effect due to Cracking and Damage*, Cachan, France (Edited by J. Mazars and Z. P. Bazant), pp. 55–64. Elsevier, London.
- Needleman, A. (1988). Material rate dependence and mesh sensitivity in localization problems. *Comp. Meth. Appl. Mech. Engng* **67**, 69–85.
- Pietruszczak, St. and Moroz, Z. (1981). Finite element analysis of deformation of strain-softening materials. *J. Numer. Meth. Engng* **17**, 327–334.
- Pijaudier-Cabot, G. and Bazant, Z. P. (1987). Nonlocal damage theory. *J. Engng Mech. ASCE* **113**, 1512–1533.

- Saouridis, C. and Mazars, J. (1988). A multiscale approach to distributed damage and its usefulness for capturing structural size effect. In *Proc. France-US Workshop on Strain Localization and Size Effect due to Cracking and Damage*, Cachan, France (Edited by J. Mazars and Z. P. Bazant), pp. 391–403. Elsevier, London.
- Schreyer, H. L. (1988). Formulations of nonlocal softening in a finite zone with anisotropic damage. In *Proc. France-US Workshop on Strain Localization and Size Effect due to Cracking and Damage*, Cachan, France (Edited by J. Mazars and Z. P. Bazant), pp. 426–439. Elsevier, London.
- Schreyer, H. L. (1990). Analytical solutions for nonlinear strain-gradient softening and localization. *J. Appl. Mech. ASME* **57**, 522–528.
- Schreyer, H. L. and Chen, Z. (1986). One-dimensional softening with localization. *J. Appl. Mech. ASME* **53**, 791–797.
- Simo, J. C. (1988). Strain softening and dissipation: a unification of approaches. In *Proc. France-US Workshop on Strain Localization and Size Effect due to Cracking and Damage*, Cachan, France (Edited by J. Mazars and Z. P. Bazant), pp. 440–461. Elsevier, London.
- Simo, J. C. and Ju, J. W. (1987). Strain- and stress-based continuum damage models—I, II. *Int. J. Solids Structures* **23**, 821–869.
- Triantafyllidis, N. and Aifantis, E. C. (1986). A gradient approach to localization of deformation. I. Hyperelastic materials. *J. Elast.* **16**, 225–237.

#### APPENDIX: GENERALIZATION TO THREE DIMENSIONS

The three-dimensional equivalent of the constitutive relationship (3) can be written as

$$\sigma(\mathbf{x}, t) = [1 - D(\mathbf{x}, t)] {}^4\mathbb{H} : \epsilon(\mathbf{x}, t), \quad (\text{A1})$$

where  $\sigma$  and  $\epsilon$  are the stress and strain tensor, respectively, with  $D$  the isotropic damage and with  ${}^4\mathbb{H}$  the stiffness tensor of rank four with elastic moduli. The vector  $\mathbf{x}$  indicates the position of a material point of the configuration. The damage equivalent strain, see (4) in the one-dimensional case, should be defined as a functional of the strain tensor:

$$\epsilon_d(\mathbf{x}, t) = \text{Max} \{ \epsilon_t : g(\epsilon(\mathbf{x}, \tau)) | \tau \leq t \}. \quad (\text{A2})$$

The operator  $g(\cdot)$ , a material dependent scalar mapping of a tensorial (strain) quantity, is assumed to satisfy  $g(\mathbb{A}) > 0$  and  $g(\alpha\mathbb{A}) = \alpha g(\mathbb{A})$  for all  $\mathbb{A} \neq \mathbb{0}$  and all  $\alpha \geq 0$ . In conformity with (5) the damage equivalent strain controls the evolution of the damage according to

$$\begin{aligned} \dot{D}(\mathbf{x}, t) &> 0 && \text{if } g(\epsilon(\mathbf{x}, t)) = \epsilon_d(\mathbf{x}, t) \quad \text{and if } \dot{g}(\epsilon(\mathbf{x}, t)) > 0, \\ \dot{D}(\mathbf{x}, t) &> 0 && \text{if } g(\epsilon(\mathbf{x}, t)) < \epsilon_d(\mathbf{x}, t) \quad \text{or if } \dot{g}(\epsilon(\mathbf{x}, t)) \leq 0. \end{aligned} \quad (\text{A3})$$

With the application of the operator  $g(\cdot)$ , thus following an associative procedure, the equivalent of (6) can be derived:

$$D(\mathbf{x}, t) = \frac{g(\epsilon(\mathbf{x}, t)) - g({}^4\mathbb{H}^{-1} : \sigma(\mathbf{x}, t))}{g(\epsilon(\mathbf{x}, t))} \quad (\text{A4})$$

and along the same lines as pursued in Sections 2 and 3, the damage evolution, to be compared to (8), can be written as:

$$\begin{aligned} D(\mathbf{x}, t) &= \frac{1}{g(\epsilon(\mathbf{x}, t))} \int \int \int_{\|\mathbf{s}\| \leq \Delta} W(\|\mathbf{s}\|) \|\mathbf{s} \cdot \nabla g(\epsilon(\mathbf{x}, t))\| dV \\ &= \frac{m}{g(\epsilon(\mathbf{x}, t))} \|\nabla g(\epsilon(\mathbf{x}, t))\| \\ &\quad \text{if } g(\epsilon(\mathbf{x}, t)) = \epsilon_d(\mathbf{x}, t) \quad \text{and if } \dot{g}(\epsilon(\mathbf{x}, t)) > 0, \\ D(\mathbf{x}, t) &= \text{Max} \{ D(\mathbf{x}, \tau) | \tau \leq t \} \\ &\quad \text{if } g(\epsilon(\mathbf{x}, t)) < \epsilon_d(\mathbf{x}, t) \quad \text{or if } \dot{g}(\epsilon(\mathbf{x}, t)) \leq 0, \end{aligned} \quad (\text{A5})$$

where the material parameter  $m$  is now defined by:

$$m = 2\pi \int_0^\Delta W(r) r^3 dr. \quad (\text{A6})$$

With the set of equations (A1), (A2) and (A5) the three-dimensional theory is formally accomplished. The constitutive description of the softening in the centre of the localization zone completes this system.